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On Geometric Enegineering of Supersymmetric Gauge Theories

Adil Belhaj*

High Energy Physics Laboratory, Faculty of sciences , Rabat , Morocco

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Abstract

We present the basis ideas of geometric engineering of the supersymmetric quantum field theories viewed as a low energy limit of type II strings and F-theory on singular Calabi Yau manifolds. We first give the main lines of toric geometry as it is a pwerful technique to deal with compact complex manifolds. Then we introduce mirror symmetry which plays a crucial role in the sudy of superstring dualities and finally we give elments on Calabi Yau singularities. After that we study the geometric engineering of $N = 2$ supersymmetric gauge theories in six and four dimensions. Finally we make comments regarding $N = 1$ SYM in four dimensions.

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1 Introduction

Over the few past years we have learned how many non trivial spuersymmetric QFT are obtained from singular limits of type II strings, M-theory and F-theory compactifications by using the geometric engineering method introduced [1] and developped by Kazt, Mayr and Vafa in [2], see also [3, 4, 5] The basic tools of this tricky method are toric geometry of ADE singularities of the K3 surface and local mirror symmetry. In this method, the complex and Khaler deforamtion parameters of the singularities are related to physical parameters in the low energy limit of string compactifications. The most familiar example is type IIA string on K3 near ADE singularities, which believed to be dual to heterotic string on T^4. In this case the low energy limit is described by supersummetric theories in six dimensions with ADE gauge groups. The moduli space of gauge invariant vacua of these models is just the moduli space of K3, which may be viewed as describing the motion of wrapped D2 brane on two cycles of K3. In this communication we focuss our attention on the study type II superstrings compactification on ADE hypersurface singularities. In particular we study the embedding the of $4d$ supersymmetric QFT in type IIA string compactification on Calabi Yau three folds near the ADE singularities. We also discuss the geometric engineering of the interersting case of $N = 1$ supersymmetric gauge theories, obtained from F-theory on singular elliptic Calabi Yau manifolds [5].

2 Toric geometry and Calabi Yau singularities

Here we review brefly some basic facts about toric geometry useful for the study of superstring compactifications [6]. Roughly speaking, toric geometry concerns n complex dimensional manifolds which can be represented by a polytope Δ of the n dimensional Z^n hypercubic lattice of R^n. Instead of using direct complex analysis methods for complex manifolds, it is interesting to apply technics of toric geometry. Toric geometry is valuabe tool for the discussion of geometric properties of Calabi Yau manifolds which are important in the context of string compatifications and Calabi Yau fiber bundles involved in F-theory - Heterotic duality.

Simple example of toric varieties are given by weigthed projetctive spaces (WCP^n). These spaces can be defined as

$$WCP^n = \frac{C^{n+1} - \vec{0}}{C^*} \tag{2.1}$$

with the C^* action

$$C^* : x_i \to \lambda^{q_i} x_i, \quad i = 1, 2, \dots, n+1. \tag{2.2}$$

The leading CP^1 example parametrized $\{(z_1, z_2)/(z_1, z_2) = (\lambda z_1, \lambda z_2), \lambda \in C^*\}$ is just the complex line which is know to be isomorphism to the real two sphere $S^2 \approx SU(2)/U(1)$. a less triavial example is given by $WCP^2(2, 3, 1)$. In this case, the equivalence relation (2.2) becomes

$$(x_1, x_2, x_3) \rightarrow (\lambda^2 x_1, \lambda^3 x_2, \lambda x_3). \tag{2.3}$$

This relation can be encoded in triangle in R^2, with the following three vertices $v_{x_1} = (-1, 0)$, $v_{x_2} = (0, -1)$ and $v_{x_3} = (2, 3)$ in Z^2, such that

$$2v_{x_1} + 3v_{x_2} + v_{x_3} = 0. \tag{2.4}$$

More general d-dimensional toric manifolds are generalizations of these weigthetd projectives spaces which are defined as

$$V_\Delta^d = \frac{C^k - U}{C^{*r}} \tag{2.5}$$

where the U set and the C^* action are given by:

$$C^{*r} : x_i \rightarrow \lambda^{q_i^a} x_i, i = 1, 2, \ldots, d + r; a = 1, 2, \ldots, r \tag{2.6}$$

$$U = \cup_I \{(x_1, \ldots x_{d+r}), \quad x_i = 0 \quad for \quad i \in I\}. \tag{2.7}$$

The toric manifold (2.6) extends the complex projective space WCP^n in the sense that instead of removing the origin, one removes the set U and takes the quotient by the C^* actions. d-dimensional toric manifolds V_Δ^d have many remarkable properties one of them is that they may be encoded in toric diagram Δ of $k = d + r$ vertices v_i embedded in the Z^d lattice such that

$$\sum_{i=1}^{d+r} q_i^a v_i = 0, \quad a = 1, \ldots, r. \tag{2.8}$$

In these eqs (2.8), the q_i^a's are the Mori vetcors defining the intersection matrix of divisors of toric manifld V_Δ^d. Note in passing that toric geometry is intimately related to $2d$ $N = 2$ supersymmetric sigma models. In the case where the target space is V_Δ^d , then the charges q_i^a are interpreted as the charges of the matter fields (x_i), and ess (2.8) is linked to the D-flatness eqs of $2d$ $N = 2$ gauge theory the describing the flat direction as shown here below

$$\sum_{i=1}^{d+r} q_i^a |x_i|^2 = R_a \tag{2.9}$$

In this eqs R_a's are the FI coupling parameters which describe Kahler parameters of toric manifolds . Note also that the first chern class of these space is proportional to $\sum\limits_{i,a} q_i^a$ [7]. For $\sum\limits_{i,a} q_i^a = 0$, the V_Δ^d becomes a Calabi Yau manifold. This means physically that the $N = 2$ syper symetric theory flows to $N = 2$ SCFT [7, 8].

Another important tool of toric geometry is mirror symmetry. The latter is a symmetry which transforms into each oyher Kahler and complex structres of complex d- dimensional Calabi Yau M and W. A mirror pair has Hodge numbers satisfying the mirror relations.

$$\begin{array}{l} h^{1,1}(M) = h^{d-1,1}(W) \\ h^{d-1,1}(M) = h^{1,1}(W), \end{array} \tag{2.10}$$

this means that the complex moduli space of M is identical to the Kahler moduli space of W and vice versa. Mirror symmetry plays a central role in the study of type II superstring compatifications and in the determination of the moduli space of vacua. This transformation can be viewed as a generalization of T duality in type II string on Calabi Yau manifold [6]. iT would be interesting to note that the mirror symmetry has played a crucial role in the developement in syperstrin and QFT dualities and in the obtention of exact results. For example, l'absence of type IIA dilaton in the vector multiplet, has been exploited to derive exact solution in the Coulomb branch of $N = 2$ QFT in four dimensions by using mirror symmetry.

2.1 ADE hypersurfaces

Toric varieties may have singularites, which are very important in the understanding the non perturbative solutions of gauge theories. Some of these singularities are given by the so called ADE singularites, which read as :

$$\begin{array}{l} A_n : xy + z^n = 0 \\ D_n : x^2 + y^2 z + z^{n-1} = 0 \\ E_6 : x^2 + y^3 + z^4 = 0 \\ E_7 : x^2 + y^3 + yz^3 = 0 \\ E_8 : x^2 + y^3 + z^5 = 0. \end{array} \tag{2.11}$$

These equations describe complex surfaces embedded in C^3 with coordinates x, y, z. Each of them has a singularity at $x = y = z = 0$. ADE singularities may be sesolved either by deforming the complex structure or the Kahler oneto obtain a smooth manifold. Kahler deformation consists to blow up the singular point by intersecting 2-spheres ranged according to the Dynkin diagram of ADE Lie algebra. The intersection matrices of the blowing up

2-spheres C_2 , of the resolution of the ADE singularities are given by the Mori vectors q_i^a (eq (2.4-5)) , which up to sign, coincides with the Cartan matrices K_{ij} of the ADE Lie algebras. This nice connection between singularites and Lie algebras plays an important role in the geometric engineering of the $N = 2$ supersymmetric quantum field theory in four dimensions obtained from type II strings compatification on local Calbi Yau threefolds [2] and in the geometric engineering of $N = 1$ models from F-theory compactification on elliptic Calabi Yau manifolds [5].

3 Geometric engineering of $N = 2$ QFT in four dimensions

Geomeetric engineering of $4dN = 2$ supersymmetric quantum field theories is a geomerical method allowing to get the relevent moduli from type IIA string compactification on local Calabi Yau threefold M_3 with ADE singularites. In this method M_3 is realized as a local K3 (ALE space) fibered over a base which may be thought of as a 2-sphere or a collection of intersecting 2-spheres. The gauge fields of the QFT is obtained from D2 branes wrapped on the 2 cycles of the singularities of the fiber $K3$ and matter is given by non trivial geometry on the base of M_3. The physical parameters of the field theory are related to the moduli space of both the fiber(F) and the base (B) of M_3. The gauge coupling g is proprotional to the inverse of the square root of the volume of the base $V(B)$, i.e

$$V(B) = g^{-2}.$$

Before giving the main steps in getting $4dN = 2$ from IIA string on M_3. Let us begin by describing type IIA compactification on local K3 with ADE singularities.

3.1 N=2 in six dimensions

Type IIA on K3 near ADE singularities give a $6d$ $N = 2$ supersymmetric gauge theory with ADE gauge symmetry. To fix the ideas suppose for simplicity that $K3$ has a $su(2)$ singularity. The local geometry of this background is described by the complex equation:

$$xy = z^2$$

Type IIA on K3 with $SU(2)$ singularity gives a $N = 2$ $SU(2)$ gauge theory in six dimensions. In this case D2-branes wrapping around the blow down 2-sphere give two $W_\mu^\pm$ massless vector particles depending of the two possible orientations for the wrapping. The $W_\mu^\pm$ gauge field are charged under the $U(1)$ gauge boson Z_0^μ obtained by decomposing the type IIA superstring 3-form in terms of the harmonic forms of the vanishing 2-sphere . Then near an A_1 singulariry of K3, we get three masslesss vector particles $W_\mu^\pm$ and Z_0^μ which altogether form an $SU(2)$ adjoint. We thus obtain a $N=2$ $SU(2)$ gauge theory in 6 dimensions. More genererally, if the single vanishing two sphere is replaced by a collection of intersectiong two sphere accordintg to the ADE Dynkin Diagrams, one get a $6d$ $N=2$ sypersymmetric gauge theory with ADE gauge group.

3.2 N=2 in four dimensions

To obtain QFT's in four dimensions, one has to consider a further compactification on a one copmlex dimensional base of M_3. If the (B) is taken that is on a 2-shpere, then one gets a $N=2$ pure $SU(2)$ Yang -Mills in 4 dimensions. To incorporate matter, we consider non trivial geometry on the base of M_3. If we have a 2 dimensional locus with $SU(n)$ singularity and another locus with $SU(m)$ singularity and they meet to a point, the mixed wrapped 2 cycles will now lead to (n,m) $N=2$ bi-fundamental matter of the $SU(n)\times SU(m)$ gauge symmetry in four dimensions. Geometrically, this means that the base geometry of M_3 is given by two intersecting P^1 spheres whose volumes V_1 and V_2 define the gauge coupling constant g_1 and g_2 of the $SU(n)$ and $SU(m)$ symmetries respectively. Note that we can also engineer the adjoint matter. Moreover if we choose the base (B) as a collection of intersectin 2-sphere according to afine Dynkin diagrams, then one engineers $N=2$ superconrmal field theories in four dimensions.

Geometric engineering of $4d$ $N=2$ QFT is really a tricky method to study $4d$ $N=2$ QFT embedded in type IIA superstring theory. In this method, $4dN=2$ QFT's are represented by quiver diagrams where for each SU gauge group factor we consider a node, and for each pair of groups with bi-fundamental matter, we connect the corresponding nodes with a line. These diagrams have a similar representation as the ADE Dynkin diagrams of ordinary and affine simply laced Lie algebras. The developments obtained over the few last years are nicely described in this approach. Geometric engineening is a powerful methgod to engineer the low enrgy of string theory especially in the study on non perturbativefs QFT's. In this regards, it is worthwhile to mention the three following due to this construction :

(i) The derivation of exact solutions of Coulomb branch of $4d$ $N=2$ QFT which are obtained by help of local mirror symmetry.

(ii) The classification of $4d$ $N=2$ superconformal theories in terms of affine ADE diagrams.

This analysis is also valid in the non simply laced cases [4]
(iii) The gauge coupling space of these superconfmal field theories is linked to the moduli of flat connections on the torus. These moduli is interseting in the study of the duality between heterotic string on elliptically fibered compact manifolds and F-theory, and in geometric construction of $N = 1$ QFT .

4 Conclusion and Discussion

We conclude this communication by discussing the geometric construction of $N = 1$ Yang - Mills in four dimensions. These models may be obtained in terms of F-theory on elliptic Callabi You fourfolds. The latter is realized as elliptically fibered K3, with affine ADE singularities, fiber over a complex base space. If we choose the base a P^2 or two complex dimensional toric space F^n , this gives $N = 1$ Yang Mills in four dimensions with ADE gauge symmetry. Morevover we can also engineer $N = 1$ models with non simply laced gauge symmetry by using the analysis of [5]. This analysis in based on toric realization of folding method of ADE Lie algebras. In that our work we have distinguuich two possible toric realization depending the action of the folding on the elliptic curve of the fiber K3.
In this work we have studied the geometric engineering of supersymmetric gauge theories obtained as a low energy limit of type II string on Calabi Yau manifold with ADE singularities. It turns out ADE singularities of K3 lead to appearance of corresponding gauge group in physics. Moreover these singularities have toric realizations, which are related to khaler Calabi Yau constraction from $2d$ $N = 2$ sigma models. Thus it is natural to think about another type of singularities and their gauge theories correspoding, such as hyperkahler singularities which are linked to $2d$ $N = 4$ sigma models approach [9, 10]. These singularities maight be used to derive new physics, not described by a conventional gauge theory.
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References

[1] S. Katz , A. Klemm and C. Vafa, " Geometric engineering of quantum field theories ," *Nucl Phys.* **B497** (1997) 173. `hep-th/9609239`.

[2] S. Katz, P. Mayr and C. Vafa, " Mirror Symmetry and Exact solution of $4dN = 2$ gauge theories ," *Adv. THeor. Math Phys.* **1** (1998) 35. `hep-th/9706110`.

[3] A. Belhaj, A. El. Fallah and E. H. Saidi ,“ On the affine D_4 mirror geometry ,” *CQG* **16** (1999) 3297 .

[4] A. Belhaj, A. EL. Fallah and E. H. Saidi , “On non simply laced miorror geometry in Type II strings ,” *CQG* **1** (2000) 515**12** (1999) 021, hep-th/9910196.

[5] A. Belhaj, and E. H. Saidi in preparation.

[6] N.C. Leung and C. Vafa,” Branes and Toric Geometry, ”hep-th/9711013.

[7] K. Hori, and C. Vafa ” Mirror Symmetry” hep-th/0002222.

[8] K. Hori, A. Iqbal and C. Vafa,” D-branes and Mirror Symmetry”, hep-th/0005247.

[9] A. Belhaj, and E. H. Saidi,” HyperKahler singularities in Superstrings Compactifiaction and $2dN = 4$ conformal Field theory ”, hep-th/ 0002205.

[10] A. Belhaj, and E. H. Saidi,” On HyperKahler Singularities ” under submission.